

## HIGHER DIMENSIONAL GRAY HERMITIAN MANIFOLDS

WŁODZIMIERZ JELONEK

ABSTRACT. The aim of this paper is to describe a large class of Hermitian Gray manifolds.

**0. Introduction.** The Gray manifolds (also called  $\mathcal{AC}^\perp$ -manifolds) are Riemannian manifolds  $(M, g)$  whose Ricci tensor  $\rho$  satisfies the condition that  $\rho - \frac{2\tau}{n+2}g$  is a Killing tensor, where  $\tau$  is the scalar curvature of  $(M, g)$ . The equivalent condition is  $\nabla_X \rho(X, X) - \frac{2X\tau}{n+2}g(X, X) = 0$  for all  $X \in TM$ . This condition was first considered by A. Gray in [G]. The class of Gray manifolds is in some sense close to Einstein manifolds. The problem of describing compact Gray manifolds was stated by Besse in [Bes]. The Einstein structures on  $S^2$ -bundles were constructed by J. Wang and M. Wang in [W-W] and Bérard Bergery in [Ber], non-homogeneous Kähler Einstein manifolds were constructed in [K-S], and conformally Kähler Einstein manifolds were classified by Derdziński and Maschler in [D-M-2]. In [J-2], [J-3] we have described the class of compact four dimensional Gray manifolds with two eigenvalues of the Ricci tensor. These manifolds were the sphere bundles over Riemannian surfaces of any genus  $g$ . In addition these manifolds were Hermitian with respect to two opposite complex structures. The Gray Kähler manifolds are weakly Bochner flat -(WBF) Kähler manifolds. The class of WBF Kähler manifolds was investigated for example in ([A-C-G], [A-C-G-T]).

The aim of the present paper is to classify compact Gray manifolds with two eigenvalues of the Ricci tensor, whose Ricci tensor is Hermitian with respect to a conformally Kähler complex structure and which have two-dimensional, integrable and totally geodesic eigendistribution of the Ricci tensor. Using the theorem of Derdziński and Maschler ([D-M-1], [D-M-2]) we prove that such structures exist on  $\mathbb{CP}^1$ -bundles over Kähler-Einstein manifolds of any dimension  $2(n-1)$ ,  $n \geq 2$ , and on  $\mathbb{CP}^n$  and give a complete classification of such metrics. We shall prove that in the case of  $\mathbb{CP}^1$ -bundles the eigenvalues of the Ricci tensor are everywhere different and our manifolds are also Hermitian with respect to an opposite oriented complex structure (we call them bi-Hermitian). In the case of  $\mathbb{CP}^n$  the eigenvalues coincide at exactly one point where the opposite Hermitian structure is not defined. In particular, we classify the class of compact Gray Kähler manifolds (i.e. WBF Kähler manifolds) whose Ricci tensor has two eigenvalues and two-dimensional, totally geodesic eigendistribution. These Kähler manifolds are also described in [A-C-G-T].

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MS Classification: 53C55, 53C25. Key words and phrases: Gray manifold, Hermitian metric, Killing tensor, special Kähler-Ricci potential

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

**1. Killing tensors.** Our notation is as in [K-N]. Let  $(M, g)$  be a smooth connected Riemannian manifold. For a tensor  $T(X_1, X_2, \dots, X_k)$  we define a tensor  $\nabla T(X_0, X_1, \dots, X_k)$  by  $\nabla T(X_0, X_1, \dots, X_k) = \nabla_{X_0} T(X_1, \dots, X_k)$ . By a Killing tensor on  $M$  we mean an endomorphism  $S \in \text{End}(TM)$  satisfying the following conditions:

$$(1.1) \quad g(SX, Y) = g(X, SY) \text{ for all } X, Y \in TM,$$

$$(1.2) \quad g(\nabla S(X, X), X) = 0 \text{ for all } X \in TM.$$

We call  $S$  a proper Killing tensor if  $\nabla S \neq 0$ . We denote by  $\Phi$  the tensor defined by  $\Phi(X, Y) = g(SX, Y)$ .

We start with :

**Proposition 1.1.** *The following conditions are equivalent:*

- (a) *A tensor  $S$  is a Killing tensor on  $(M, g)$ ;*
- (b) *For every geodesic  $\gamma$  on  $(M, g)$  the function  $\Phi(\gamma'(t), \gamma'(t))$  is constant on  $\text{dom} \gamma$ .*
- (c) *The condition*

$$(A) \quad \nabla_X \Phi(Y, Z) + \nabla_Z \Phi(X, Y) + \nabla_Y \Phi(Z, X) = 0$$

*is satisfied for all  $X, Y, Z \in \mathfrak{X}(M)$ .*

*Proof:* By using polarization it is easy to see that (a) is equivalent to (c). Let now  $X \in T_{x_0} M$  be any vector and  $\gamma$  be a geodesic satisfying the initial condition  $\gamma'(0) = X$ . Then

$$(1.3) \quad \frac{d}{dt} \Phi(\gamma'(t), \gamma'(t)) = \nabla_{\gamma'(t)} \Phi(\gamma'(t), \gamma'(t)).$$

Hence  $\frac{d}{dt} \Phi(\gamma'(t), \gamma'(t))_{t=0} = \nabla \Phi(X, X, X)$ . The equivalence (a)  $\Leftrightarrow$  (b) follows immediately from the above relations.  $\diamond$

As in [D] define  $E_S(x)$  to be the number of distinct eigenvalues of  $S_x$ , and set  $M_S = \{x \in M : E_S \text{ is constant in a neighbourhood of } x\}$ . The set  $M_S$  is open and dense in  $M$  and the eigenvalues  $\lambda_i$  of  $S$  are distinct and smooth in each component  $U$  of  $M_S$ . The eigenspaces  $D_\lambda = \ker(S - \lambda I)$  form smooth distributions in each component  $U$  of  $M_S$ . By  $\nabla f$  we denote the gradient of a function  $f$  ( $g(\nabla f, X) = df(X)$ ) and by  $\Gamma(D_\lambda)$  (resp.  $\mathfrak{X}(U)$ ) the set of all local sections of the bundle  $D_\lambda$  (resp. of all local vector fields on  $U$ ). Let us note that if  $\lambda \neq \mu$  are eigenvalues of  $S$  then  $D_\lambda$  is orthogonal to  $D_\mu$ .

**Theorem 1.2.** *Let  $S$  be a Killing tensor on  $M$  and  $U$  be a component of  $M_S$  and  $\lambda_1, \lambda_2, \dots, \lambda_k \in C^\infty(U)$  be eigenfunctions of  $S$ . Then for all  $X \in D_{\lambda_i}$  we have*

$$(1.4) \quad \nabla S(X, X) = -\frac{1}{2} \nabla \lambda_i \|X\|^2$$

*and  $D_{\lambda_i} \subset \ker d\lambda_i$ . If  $i \neq j$  and  $X \in \Gamma(D_{\lambda_i}), Y \in \Gamma(D_{\lambda_j})$  then*

$$(1.5) \quad g(\nabla_X X, Y) = \frac{1}{2} \frac{Y \lambda_i}{\lambda_j - \lambda_i} \|X\|^2.$$

*Proof.* Let  $X \in \Gamma(D_{\lambda_i})$  and  $Y \in \mathfrak{X}(U)$ . Then we have  $SX = \lambda_i X$  and

$$(1.6) \quad \nabla S(Y, X) + (S - \lambda_i I)(\nabla_Y X) = (Y \lambda_i)X$$

and consequently,

$$(1.7) \quad g(\nabla S(Y, X), X) = (Y \lambda_i) \|X\|^2.$$

Taking  $Y = X$  in (1.7) we obtain  $0 = X \lambda_i \|X\|^2$  by (1.2). Hence  $D_{\lambda_i} \subset \ker d\lambda_i$ . Thus from (1.6) it follows that  $\nabla S(X, X) = (\lambda_i I - S)(\nabla_X X)$ . Condition (A) implies  $g(\nabla S(X, Y), Z) + g(\nabla S(Z, X), Y) + g(\nabla S(Y, Z), X) = 0$  hence,

$$(1.8) \quad 2g(\nabla S(X, X), Y) + g(\nabla S(Y, X), X) = 0.$$

Thus, (1.8) yields  $Y \lambda_i \|X\|^2 + 2g(\nabla S(X, X), Y) = 0$ . Consequently,  $\nabla S(X, X) = -\frac{1}{2} \nabla \lambda_i \|X\|^2$ . Let now  $Y \in \Gamma(D_{\lambda_j})$ . Then we have

$$(1.9) \quad \nabla S(X, Y) + (S - \lambda_j I)(\nabla_X Y) = (X \lambda_j)Y.$$

It is also clear that  $g(\nabla S(X, X), Y) = g(\nabla S(X, Y), X) = (\lambda_j - \lambda_i)g(\nabla_X Y, X)$ . Thus,

$$Y \lambda_i \|X\|^2 = -2(\lambda_j - \lambda_i)g(\nabla_X Y, X) = 2(\lambda_j - \lambda_i)g(Y, \nabla_X X)$$

and (1.5) holds.  $\diamond$

*Corollary 1.3.* Let  $S, U, \lambda_1, \lambda_2, \dots, \lambda_k$  be as above and  $i \in \{1, 2, \dots, k\}$ . Then the following conditions are equivalent:

- (a) For all  $X \in \Gamma(D_{\lambda_i})$ ,  $\nabla_X X \in D_{\lambda_i}$ .
- (b) For all  $X, Y \in \Gamma(D_{\lambda_i})$ ,  $\nabla_X Y + \nabla_Y X \in D_{\lambda_i}$ .
- (c) For all  $X \in \Gamma(D_{\lambda_i})$ ,  $\nabla S(X, X) = 0$ .
- (d) For all  $X, Y \in \Gamma(D_{\lambda_i})$ ,  $\nabla S(X, Y) + \nabla S(Y, X) = 0$ .
- (e)  $\lambda_i$  is a constant eigenvalue of  $S$ .

Let us note that if  $X, Y \in \Gamma(D_{\lambda_i})$  then  $(D_{\lambda_i} \subset \ker d\lambda_i !)$

$$(1.10) \quad \nabla S(X, Y) - \nabla S(Y, X) = (\lambda_i I - S)([X, Y]),$$

hence the distribution  $D_{\lambda_i}$  is integrable if and only if  $\nabla S(X, Y) = \nabla S(Y, X)$  for all  $X, Y \in \Gamma(D_{\lambda_i})$ . Consequently, we obtain

*Corollary 1.4.* Let  $\lambda_i \in C^\infty(U)$  be an eigenvalue of an  $\mathcal{A}$ -tensor  $S$ . Then on  $U$  the following conditions are equivalent:

- (a)  $D_{\lambda_i}$  is integrable and  $\lambda_i$  is constant.
- (b) For all  $X, Y \in \Gamma(D_{\lambda_i})$ ,  $\nabla S(X, Y) = 0$ .
- (c)  $D_{\lambda_i}$  is autoparallel.

*Proof.* This follows from (1.4), (1.10), Corollary 1.3 and the relation  $\nabla_X Y = \nabla_Y X + [X, Y]$ .  $\diamond$

Now we shall characterize Killing tensors with two eigenvalues. We start with:

**Lemma.** Let  $S$  be a self-adjoint tensor on  $(M, g)$  with exactly two eigenvalues  $\lambda, \mu$ . If the distributions  $D_\lambda, D_\mu$  are both umbilical,  $\nabla \lambda \in \Gamma(D_\mu), \nabla \mu \in \Gamma(D_\lambda)$

and the mean curvatures  $\xi_\lambda, \xi_\mu$  of the distributions  $D_\lambda, D_\mu$  respectively satisfy the equations

$$\xi_\lambda = \frac{1}{2(\mu - \lambda)} \nabla \lambda, \quad \xi_\mu = \frac{1}{2(\lambda - \mu)} \nabla \mu,$$

then  $S$  is a Killing tensor.

By  $\rho$  we shall denote the Ricci tensor of  $(M, g)$  and by  $\tau = \text{tr}_g \rho$  the scalar curvature of  $(M, g)$ .

*Definition.* A Riemannian manifold  $(M, g)$  will be called a Gray  $\mathcal{AC}^\perp$  manifold if the tensor  $\rho - \frac{2\tau}{n+2}g$  is a Killing tensor.

In this paper Gray  $\mathcal{AC}^\perp$  manifolds will be called for short Gray manifolds or  $\mathcal{AC}^\perp$  manifolds.

**Proposition 1.** *Let  $(M, g)$  be a  $2n$ -dimensional Riemannian manifold whose Ricci tensor  $\rho$  has two eigenvalues  $\lambda_0(x), \mu_0(x)$  of multiplicity 2 and  $2(n-1)$  respectively at every point  $x$  of  $M$ . Assume that the eigendistribution  $\mathcal{D}_\lambda = \mathcal{D}$  corresponding to  $\lambda_0$  is a totally geodesic foliation and the eigendistribution  $\mathcal{D}_\mu = \mathcal{D}^\perp$  corresponding to  $\mu_0$  is umbilical. Then  $(M, g)$  is a Gray manifold if and only if  $\lambda_0 - 2\mu_0$  is constant and  $\nabla \tau \in \Gamma(\mathcal{D})$ .*

*Proof.* Let  $S_0$  be the Ricci endomorphism of  $(M, g)$ , i.e.  $\rho(X, Y) = g(S_0 X, Y)$ . Let  $S$  be the tensor defined by the formula

$$(1.11) \quad S_0 = S + \frac{\tau}{n+1} \text{id}.$$

Then

$$(1.12) \quad \text{tr } S = -\frac{(n-1)\tau}{n+1}.$$

Let  $\lambda_0, \mu_0$  be the eigenfunctions of  $S_0$  and let us assume that

$$(1.13) \quad \lambda_0 - 2\mu_0 = \frac{n+1}{n-1} C$$

where  $C \in \mathbb{R}$ . Note that  $S$  also has two eigenfunctions which we denote by  $\lambda, \mu$  respectively. It is easy to see that  $\lambda = C, \mu = -\frac{\tau}{2(n+1)} - \frac{C}{n-1}$  and  $\lambda_0 = \frac{\tau}{n+1} + C, \mu_0 = \frac{\tau}{2(n+1)} - \frac{C}{n-1}$ . Since the distribution  $\mathcal{D}^\perp$  is umbilical we have  $\nabla_X X|_{\mathcal{D}} = g(X, X)\xi$  for any  $X \in \Gamma(\mathcal{D}^\perp)$  where  $\xi$  is the mean curvature normal of  $\mathcal{D}^\perp$ . Since the distribution  $\mathcal{D}$  is totally geodesic we also have  $\nabla_X X|_{\mathcal{D}^\perp} = 0$  for any  $X \in \Gamma(\mathcal{D})$ . Let  $\{E_1, E_2, E_3, E_4, \dots, E_{2n-1}, E_{2n}\}$  be a local orthonormal basis of  $TM$  such that  $\mathcal{D} = \text{span}\{E_1, E_2\}$  and  $\mathcal{D}^\perp = \text{span}\{E_3, E_4, \dots, E_{2n}\}$ . Then  $\nabla_{E_i} E_i|_{\mathcal{D}} = \xi$  for  $i \in \{3, 4, \dots, 2n\}$ . Consequently (note that  $\nabla \mu|_{\mathcal{D}^\perp} = 0$  if and only if  $\nabla \tau|_{\mathcal{D}^\perp} = 0$ ),

$$(1.14) \quad \begin{aligned} \text{tr}_g \nabla S &= \sum_{i=3}^n \nabla S(E_i, E_i) = -2(n-1)(S - \mu \text{id})(\nabla_{E_3} E_3) + \nabla \mu|_{\mathcal{D}^\perp} \\ &= -2(n-1)(\lambda - \mu)\xi \end{aligned}$$

if we assume that  $\nabla\tau|_{\mathcal{D}^\perp} = 0$ . On the other hand,  $\text{tr}_g \nabla S_0 = \frac{\nabla\tau}{2}$  and  $\text{tr}_g \nabla S = \text{tr}_g \nabla S_0 - \frac{\nabla\tau}{n+1}$ . Consequently,

$$(1.15) \quad \text{tr}_g \nabla S = \frac{(n-1)\nabla\tau}{2(n+1)} = -(n-1)\nabla\mu.$$

Thus  $\xi = -\frac{1}{2(\mu-\lambda)}\nabla\mu$ . From the Lemma it follows that  $(M, g)$  is an  $\mathcal{AC}^\perp$ -manifold if  $\lambda_0 - 2\mu_0$  is constant and  $\nabla\tau \in \Gamma(\mathcal{D})$ . These conditions are also necessary since  $\nabla\lambda = 0$  if  $(M, g)$  is an  $\mathcal{AC}^\perp$ -manifold and  $D_\lambda$  is totally geodesic. Analogously  $\xi = -\frac{1}{2(\mu-\lambda)}\nabla\mu$  and  $\nabla\mu = -\frac{1}{2(n+1)}\nabla\tau \in \Gamma(\mathcal{D})$ , where  $\xi$  is the mean curvature normal of the umbilical distribution  $D_\mu$ , if  $(M, g)$  is an  $\mathcal{AC}^\perp$ -manifold.  $\diamond$

**2.  $\mathcal{AC}^\perp$  Hermitian manifolds with Hermitian Ricci tensor and two eigenvalues of the Ricci tensor.** In this section we classify compact  $\mathcal{AC}^\perp$  manifolds  $(M, g, J)$  with two eigenvalues of the Ricci tensor whose Ricci tensor is Hermitian with respect to a conformally Kähler complex structure  $J$ . We shall additionally assume that  $\dim D_\lambda = 2$  in the set  $U = \{x \in M : E(x) = 2\}$  where  $D_\lambda$  is defined, and that the distribution  $D_\lambda$  is integrable and totally geodesic. This is equivalent to the fact that the tensor  $S$  has a constant eigenvalue  $\lambda$  and  $D_\lambda$  is integrable. Note that the set  $U$  is open and denote by  $V$  the interior of  $\{x : E(x) = 1\}$ . Then  $U \cup V$  is an open dense subset of  $M$ . It is also clear that  $(V, g)$  is an Einstein manifold. We shall later prove that in fact  $V = \emptyset$  and either  $U = M$  or  $M = \mathbb{CP}^n$ .

Let  $f \in C^\infty(M)$  be a positive function such that the manifold  $(M, f^2g, J)$  is a Kähler manifold. This means that the metric  $g_1 = f^2g$  is Kählerian with respect to  $J$ . By  $\nabla, \nabla^1$  we shall denote the Levi-Civita connections of the metrics  $g, g_1$  respectively, and by  $\nabla F, \nabla^1 F$  the gradients of a function  $F$  with respect to the metrics  $g, g_1$ . Note that

$$(2.1) \quad \nabla^1 F = f^{-2}\nabla F,$$

and also

$$(2.2) \quad \nabla_X Y = \nabla_X^1 Y - d \ln f(X)Y - d \ln f(Y)X + g_1(X, Y)\nabla^1 \ln f.$$

Note that  $\nabla_X = \nabla_X^1 + K_X$  where  $K_X(Y) = -d \ln f(X)Y - d \ln f(Y)X + g_1(X, Y)\nabla^1 \ln f$ . Now we compute the covariant derivative  $\nabla J$  of the complex structure  $J$ :

$$(2.3) \quad \begin{aligned} \nabla_X J(Y) &= [K_X, J]Y = -d \ln f(JY)X + d \ln f(Y)JX \\ &\quad + g_1(X, JY)\nabla^1 \ln f - g_1(X, Y)J\nabla^1 \ln f. \end{aligned}$$

Note that  $g_1(X, Y)\nabla^1 \ln f = g(X, Y)\nabla \ln f$ . From (2.3) it is clear that

$$(2.4) \quad \nabla_X J(X) = -d \ln f(JX)X + d \ln f(X)JX - g_1(X, X)J\nabla^1 \ln f.$$

In particular,

$$(2.5) \quad \text{tr}_g J\nabla_X J(X) = 2(n-1)\nabla \ln f.$$

Let  $\rho, \rho_1$  be the Ricci tensors of  $(M, g), (M, g_1)$  respectively. Then

$$(2.6) \quad \rho = \rho_1 + (2n-2)\frac{1}{f}H^f - \left(\frac{1}{f}\Delta f + (2(n-1))\frac{1}{f^2}\|\nabla^1 f\|^2\right)g_1,$$

where  $H^f(X, Y) = g_1(\nabla_X^1 \nabla^1 f, Y)$  is the Hessian of  $F$  with respect to the metric  $g_1$ . Since both  $\rho, \rho_1$  are  $J$ -invariant it follows that the field  $\xi = J(\nabla^1 f)$  is a holomorphic Killing vector field for both  $(M, g, J)$  and  $(M, g_1, J)$  (see [D]). Now we prove that  $\xi \in \Gamma(D_\lambda)$ . Since  $D_\lambda$  is totally geodesic it follows that for any  $X, Y \in D_\lambda$  we have  $\nabla S(X, Y) = 0$ . On the other hand,  $S \circ J = J \circ S$  and

$$(2.7) \quad \nabla_X J \circ S + J \circ \nabla_X S = \nabla_X S \circ J + S \circ \nabla_X J.$$

Consequently, for  $X \in D_\lambda$  we get

$$(2.8) \quad \lambda \nabla J(X, X) + J \circ \nabla S(X, X) = \nabla S(X, JX) + S \circ \nabla J(X, X).$$

Thus

$$(2.9) \quad (S - \lambda I)(\nabla J(X, X)) = J(\nabla S(X, X)) = -\frac{1}{2}J\nabla\lambda = 0.$$

From (2.4) we get  $\nabla J(X, X)_\mu = 0$  where  $Y_\mu$  denotes the  $D_\mu$ -component of a vector with respect to the decomposition  $TM = D_\lambda \oplus D_\mu$ . It is clear in view of (2.4) that  $\nabla J(X, X)_\mu = -g(X, X)(J\nabla \ln f)_\mu$  and thus  $(\nabla f)_\mu = 0$ . Thus  $\xi \in \Gamma(D_\lambda)$ .

Set  $TX = \nabla_X \xi$ . Now, since  $D_\lambda$  is totally geodesic it is clear that  $TD_\lambda \subset D_\lambda$  and  $TD_\mu \subset D_\mu$ . Let  $T^1 = \nabla^1 \xi$ . Then also  $T^1 D_\lambda \subset D_\lambda$  and  $T^1 D_\mu \subset D_\mu$ . In fact,  $\nabla_X^1 \xi = TX + d \ln f(X)\xi - g(X, \xi)\nabla \ln f$ .

Now we shall prove that  $JTX = \phi X$  for all  $X \in D_\mu$  for a certain function  $\phi$ . Note that

$$(2.10) \quad \nabla S(X, \xi) = (\lambda I - S)TX = (\lambda - \mu)TX$$

if  $X \in D_\mu$ , and consequently

$$(2.11) \quad (\lambda - \mu)g(TX, JX) = g(\nabla S(X, \xi), JX) = g(\nabla S(X, JX), \xi).$$

Note that in view of (2.4) and (1.5) the following equality holds:

$$(2.12) \quad \begin{aligned} \nabla S(X, JX) &= (\mu I - S)(\nabla J(X, X) + J(\nabla_X X)) \\ &= (\mu - \lambda)(-g(X, X)J(\nabla \ln f) + g(X, X)\frac{1}{2(\lambda - \mu)}J(\nabla \mu)). \end{aligned}$$

It follows that

$$(2.13) \quad g(TX, JX) = -g(JTX, X) = g(X, X)\left(-d \ln f(J\xi) + \frac{1}{2(\lambda - \mu)}d\mu(J\xi)\right).$$

Thus for  $X \in D_\mu$ ,

$$(2.14) \quad JTX = \left(d \ln f(J\xi) - \frac{1}{2(\lambda - \mu)}d\mu(J\xi)\right)X.$$

Let us recall the definition of a special Kähler-Ricci potential ([D-M-1], [D-M-2]).

*Definition.* A nonconstant function  $\tau \in C^\infty(M)$ , where  $(M, g, J)$  is a Kähler manifold, is called a special Kähler-Ricci potential if the field  $X = J(\nabla\tau)$  is a Killing vector field and at every point with  $d\tau \neq 0$  all nonzero tangent vectors orthogonal to the fields  $X, JX$  are eigenvectors of both  $\nabla d\tau$  and the Ricci tensor  $\rho$  of  $(M, g, J)$ .

We now show that the function  $f$  is a special Kähler-Ricci potential on the Kähler manifold  $(M, g_1, J)$ . In fact, for  $X \in D_\mu$  we have

$$\begin{aligned} \nabla_X^1 \nabla^1 f &= -J(\nabla_X^1 \xi) \\ &= -J(\nabla_X \xi + d\ln f(X)\xi + d\ln f(\xi)X - g(X, \xi)\nabla \ln f) \\ &= -J(\nabla_X \xi) = -JTX = -\left(d\ln f(J\xi) - \frac{1}{2(\lambda - \mu)}d\mu(J\xi)\right)X \end{aligned}$$

and it is clear that  $D_\mu$  is an eigendistribution of  $\nabla^1 df$ . Since  $D_\mu$  is also an eigendistribution of  $\rho$  it follows from (2.6) that  $D_\mu$  is also an eigendistribution of the Ricci tensor  $\rho_1$  of  $(M, g_1, J)$ . Thus on the whole of  $U$  the function  $f$  is a special Kähler-Ricci potential. The fact that  $f$  is a special Kähler-Ricci potential on  $V$  follows from the results of Derdziński and Maschler ([D-M-1], [D-M-2]) note that  $(V, g_1, J)$  is conformally Einstein. Thus  $f$  is a special Kähler-Ricci potential on the open and dense subset  $U \cup V$  of  $M$  and consequently is a special Kähler-Ricci potential on the whole of  $M$ . If  $(M, g, J)$  is a Kähler manifold then it is a WBF manifold (see [A-C-G-T]) and consequently it is extremal. We give a short proof in this case for a completeness and for the convenience of the reader (for another proof of this fact see [A-C-G]). It follows that  $J(\nabla\tau)$  is a holomorphic Killing vector field where  $\tau$  is the scalar curvature of  $(M, g, J)$ . From our assumptions it is clear that  $\nabla\tau \in \Gamma(D_\lambda)$ . Thus

$$(2.15) \quad \nabla S(X, \nabla\tau) + (S - \lambda I)(\nabla_X \tau) = 0.$$

On the other hand, note that every such Kähler metric satisfies (see [J-1])

$$(2.16) \quad \begin{aligned} \nabla_X \rho(Y, Z) &= \frac{1}{2\dim M + 4}(g(X, Y)Z\tau + g(X, Z)Y\tau + 2g(Y, Z)X\tau \\ &\quad - g(JX, Y)(JZ)\tau - g(JX, Z)(JY)\tau), \end{aligned}$$

and consequently

$$(2.17) \quad \nabla S(X, Y) = \frac{1}{4(n+1)}(g(X, Y)\nabla\tau + Y\tau X - 2YX\tau + g(JX, Y)J\nabla\tau - (JY\tau)JX)$$

and  $\nabla S(X, \tau) = \frac{1}{4n+4}(-X\tau\nabla\tau + |\nabla\tau|^2 X + (JX\tau)J\nabla\tau)$ . Thus if  $X \in D_\mu$  we have

$$(\mu - \lambda)(\nabla_X \nabla\tau)_\mu = \frac{1}{4n+4}|\nabla\tau|^2 X.$$

Since  $D_\lambda$  is totally geodesic and  $\nabla\tau \in D_\lambda$  it follows that  $\nabla_X \nabla\tau \in D_\mu$  if  $X \in D_\mu$  (note that  $g(\nabla_X \nabla\tau, Y) = g(\nabla_Y \nabla\tau, X) = H^\tau(X, Y)$ ). Consequently, for  $X \in D_\mu$  we get

$$(2.18) \quad \nabla_X \nabla\tau = \frac{1}{(4n+4)(\mu - \lambda)}|\nabla\tau|^2 X,$$

which means that  $\tau$  is a special Kähler-Ricci potential.

From the results of Derdziński and Maschler ([D-M-1], [D-M-2]) it follows that  $(M, g, J)$  is in both cases a  $\mathbb{CP}^1$ -bundle over a Kähler Einstein manifold  $N$  or is  $\mathbb{CP}^n$  in the non-Kähler case. These  $\mathbb{CP}^1$ -bundles are  $\mathbb{P}(L \oplus \mathcal{O})$  where  $L \rightarrow N$  is a complex line bundle whose Euler class is proportional to the Kähler class of  $N$  and  $\mathcal{O}$  is the trivial line bundle.

**3. Construction of  $\mathcal{AC}^\perp$  Hermitian manifolds.** In our construction we shall follow L. Bérard Bergery (see [Ber], [S]). Let  $(N, h, J)$  be a compact Kähler Einstein manifold, which is not Ricci flat and  $\dim N = 2m$ ,  $s \geq 0$ ,  $L > 0$ ,  $s \in \mathbb{Q}$ ,  $L \in \mathbb{R}$ , and  $g : [0, L] \rightarrow \mathbb{R}$  be a positive, smooth function on  $[0, L]$  which is even at 0 and  $L$ , i.e. there exists an  $\epsilon > 0$  and even, smooth functions  $g_1, g_2 : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$  such that  $g(t) = g_1(t)$  for  $t \in [0, \epsilon)$  and  $g(t) = g_2(L - t)$  for  $t \in (L - \epsilon, L]$ . Let  $f : (0, L) \rightarrow \mathbb{R}$  be positive on  $(0, L)$ ,  $f(0) = f(L) = 0$  and let  $f$  be odd at the points 0,  $L$ . Let  $P$  be a circle bundle over  $N$  classified by the integral cohomology class  $\frac{s}{2}c_1(N) \in H^2(N, \mathbb{R})$  if  $c_1(M) \neq 0$ . Let  $q$  be the unique positive integer such that  $c_1(N) = q\alpha$  where  $\alpha \in H^2(N, \mathbb{R})$  is an indivisible integral class. Such a  $q$  exists if  $N$  is simply connected or  $\dim N = 2$ . Note that every Kähler Einstein manifold with positive scalar curvature is simply connected. Then

$$s = \frac{2k}{q}; k \in \mathbb{Z}.$$

It is known that  $q = n$  if  $N = \mathbb{CP}^{n-1}$  (see [Bes], p.273). Note that  $c_1(N) = \{\frac{1}{2\pi}\rho_N\} = \{\frac{\tau_N}{4m\pi}\Omega_N\}$  where  $\rho_N = \frac{\tau_N}{2m}\Omega_N$  is the Ricci form of  $(N, h, J)$ ,  $\tau_N$  is the scalar curvature of  $(N, h)$  and  $\Omega_N$  is the Kähler form of  $(N, h, J)$ . We can assume that  $\tau_N = \pm 4m$ . In the case  $c_1(N) = 0$  we shall assume that  $(N, h, J)$  is a Hodge manifold, i.e. the cohomology class  $\{\frac{s}{2\pi}\Omega_N\}$  is an integral class. On the bundle  $p : P \rightarrow N$  there exists a connection form  $\theta$  such that  $d\theta = sp^*\Omega_N$  where  $p : P \rightarrow N$  is the bundle projection. Let us consider the manifold  $U_{s,f,g} = (0, L) \times P$  with the metric

$$(3.1) \quad g = dt^2 + f(t)^2\theta^2 + g(t)^2p^*h.$$

It is known that the metric (3.1) extends to a metric on the sphere bundle  $M = P \times_{S^1} \mathbb{CP}^1$  if and only if a function  $g$  is positive and smooth on  $[0, L]$ , even at the points 0,  $L$ , the function  $f$  is positive on  $(0, L)$ , smooth and odd at 0,  $L$  and additionally

$$(3.2) \quad f'(0) = 1, \quad f'(L) = -1$$

Then the metric (3.1) is bi-Hermitian. We shall prove this in Section 4.

The metric (3.1) extends to a metric on  $\mathbb{CP}^n$  if and only if the function  $g$  is positive and smooth on  $[0, L]$ , even at 0, odd at  $L$ , the function  $f$  is positive, smooth and odd at 0,  $L$  and additionally

$$(3.3) \quad f'(0) = 1, \quad f'(L) = -1, \quad g(L) = 0, \quad g'(L) = -1.$$

**4. Circle bundles.** Let  $(N, h, J)$  be a Kähler manifold with integral class  $\{\frac{1}{\pi}\Omega_N\}$  and let  $p : P \rightarrow N$  be a circle bundle with a connection form  $\theta$  such that



$d\theta = s\Omega_N$ , where  $s \in \mathbb{Q}$  (see [K]). Let us consider a Riemannian metric  $g$  on  $P$  where

$$(4.1) \quad g = a^2\theta \otimes \theta + b^2p^*h$$

where  $a, b \in \mathbb{R}$ . Let  $\xi$  be a fundamental vector field of the action of  $S^1$  on  $P$ , i.e.  $\theta(\xi) = 1, L_\xi g = 0$ . It follows that  $\xi \in \mathfrak{iso}(P)$  and  $a^2\theta = g(\xi, \cdot)$ . Consequently,

$$(4.2) \quad a^2 d\theta(X, Y) = 2g(TX, Y)$$

for every  $X, Y \in \mathfrak{X}(P)$  where  $TX = \nabla_X \xi$ . Note that  $g(\xi, \xi) = a^2$  is constant, hence  $T\xi = 0$ . On the other hand,  $d\theta(X, Y) = sp^*\Omega_N(X, Y) = sh(Jp(X), p(Y))$ . Note that there exists a tensor field  $\tilde{J}$  on  $P$  such that  $\tilde{J}\xi = 0$  and  $\tilde{J}(X) = (JX_*)^H$  where  $X = X_*^H$  is the horizontal lift of  $X_*$ . Indeed,  $L_\xi T = 0$  and  $T\xi = 0$ , hence  $T$  is a horizontal lift of the tensor  $\tilde{T}$ . Now  $\tilde{J} = \frac{2b^2}{sa^2}\tilde{T}$ . Since  $T\xi = 0$  we get  $\nabla T(X, \xi) + T^2X = 0$  and  $R(X, \xi)\xi = -T^2X$ . Thus  $g(R(X, \xi)\xi, X) = \|TX\|^2$  and

$$\rho(\xi, \xi) = \|T\|^2 = \frac{sa^4}{4b^4}2m.$$

Consequently,

$$(4.3) \quad \lambda = \rho\left(\frac{\xi}{a}, \frac{\xi}{a}\right) = \frac{1}{a^2}\|T\|^2 = \frac{s^2a^2}{4b^4}m.$$

We shall compute the O'Neill tensor  $A$  (see [ON]) of the Riemannian submersion  $p : (P, g) \rightarrow (N, b^2h)$ . We have

$$A_E F = \mathcal{V}(\nabla_{\mathcal{H}E} \mathcal{H}F) + \mathcal{H}(\nabla_{\mathcal{H}E} \mathcal{V}F).$$

Let us write  $u = \mathcal{V}(\nabla_{\mathcal{H}E} \mathcal{H}F)$  and  $v = \mathcal{H}(\nabla_{\mathcal{H}E} \mathcal{V}F)$ . The vertical component of a field  $E$  equals  $\theta(E)\xi$ . If  $X, Y \in \mathcal{H}$  then

$$(4.4) \quad g\left(\nabla_X Y, \frac{1}{a}\xi\right) = \frac{1}{a}(Xg(Y, \xi) - g(Y, \nabla_X \xi)) = -\frac{1}{a}g(TX, Y) = \frac{1}{a}g(X, TY).$$

Hence  $u = \frac{1}{a}g(E - \theta(E)\xi)$  and  $T(F - \theta(F)\xi)\frac{\xi}{a} = \frac{1}{a^2}g(E, TF)\xi$ . Note that  $\mathcal{H}(\nabla_X f\xi) = f\mathcal{H}(\nabla_X \xi) = fTX$ , hence

$$v = \mathcal{H}(\nabla_{\mathcal{H}E} \mathcal{V}F) = \theta(F)T(E) = \frac{1}{a^2}g(\xi, F)TE.$$

Consequently,

$$(4.5) \quad A_E F = \frac{1}{a^2}(g(E, TF)\xi + g(\xi, F)TE).$$

If  $U, V \in \mathcal{H}$  then

$$\|A_U V\|^2 = \frac{1}{a^2}g(E, TF)^2 = \frac{s^2a^2}{4b^4}g(E, \tilde{J}F)^2.$$

If  $E$  is horizontal and  $F$  is vertical then

$$(4.6) \quad A_E F = \frac{1}{a^2} g(\xi, F) T E.$$

Hence  $A_E \xi = T E$  and  $\|A_E \xi\|^2 = \|T E\|^2 = \frac{s^2 a^4}{4b^4}$ . It follows that

$$K(P_{E\xi}) = \frac{s^2 a^2}{4b^4},$$

where  $K(P_{EF})$  denotes the sectional curvature of the plane generated by the vectors  $E, F$ . If  $E, F \in \mathcal{H}$  then

$$K(P_{EF}) = K_*(P_{E_* F_*}) - \frac{3g(E, TF)^2}{a^2 \|E \wedge F\|^2},$$

where  $E_*$  means the projection of  $E$  on  $M$ , i.e.  $E_* = p(E)$ . Thus

$$(4.7) \quad K(P_{EF}) = \frac{1}{b^2} K_0(P_{E_* F_*}) - \frac{3s^2 a^2 g(E, \tilde{J}F)^2}{4b^4 \|E \wedge F\|^2},$$

where  $K_0$  means the sectional curvature of the metric  $h$  on  $N$ . Applying this we get for any  $E \in \mathcal{H}$  the formula for the Ricci tensor  $\rho$  of  $(M, g)$ :

$$\rho(E, E) = \frac{1}{b^2} \rho_0(bE_*, bE_*) - \frac{3s^2 a^2}{4b^4} + \frac{s^2 a^2}{4b^4},$$

where  $\rho_0$  is the Ricci tensor of  $(M, h)$ . Hence

$$\mu = \frac{\mu_0}{b^2} - \frac{s^2 a^2}{2b^4},$$

where  $\rho_0 = \mu_0 g_0$ . Now we shall find a formula for  $R(X, \xi)Y$  where  $X, Y \in \mathcal{H}$ . We have  $R(X, \xi)Y = \nabla T(X, Y)$  and

$$(4.8) \quad \begin{aligned} \nabla T(X, Y) &= \nabla_X(T(Y)) - T(\nabla_X Y) = \nabla_{X_*}^*(\tilde{T}Y^*) + \frac{1}{2}\mathcal{V}[X, TY] \\ &\quad - (\tilde{T}(\nabla_{X_*}^* Y_*))^* = \frac{1}{2}\mathcal{V}[X, TY] - \frac{1}{2}sp^*\Omega_N(X, TY)\xi = -\frac{s^2 a^2}{4b^2}h(X_*, Y_*)\xi. \end{aligned}$$

Consequently,  $R(X, Y, Z, \xi) = 0$  for  $X, Y, Z \in \mathcal{H}$ , and

$$(4.9) \quad R(X, \xi, Y, \xi) = -\frac{s^2 a^4}{4b^2}h(X_*, Y_*).$$

**5. Riemannian submersion**  $p : (0, L) \times P \rightarrow (0, L)$ . In this case the O'Neill tensor  $A$  is 0. We shall compute the O'Neill tensor  $T$  (see [ON]). Let  $H = \frac{d}{dt}$  be the horizontal vector field for this submersion and  $\mathcal{D}$  be the distribution spanned by the vector fields  $H, \xi$ . If  $U, V \in \mathcal{V}$  and  $g(U, V) = 0$  then  $T(U, V) = 0$ . We also have for  $U \in \mathcal{V}$  with  $g(U, \xi) = 0$  and  $U = U_*^*$  where  $h(U_*, U_*) = 1$ ,

$$(5.1) \quad T(U, U) = -gg'H.$$

In fact,  $2g(\nabla_U V, H) = -Hg(U, V) = -2gg'h(U_*, V_*)$  if  $U = V$  and 0 if  $g(U, V) = 0$ . We also have

$$(5.2) \quad T(\xi, \xi) = -ff'H.$$

Now we shall prove that the almost complex structure  $J = J_\epsilon$  defined by

$$JH = \epsilon \frac{1}{f} \xi, JX = (J_* X_*)^* \quad \text{for } X = (X_*)^* \in \mathcal{E} = \mathcal{D}^\perp$$

where  $X_* \in TN, \epsilon \in \{-1, 1\}$ , is Hermitian with respect to the metric  $g$ . For horizontal lifts  $X, Y \in \mathfrak{X}(P) \subset \mathfrak{X}((0, L) \times P)$  of the fields  $X_*, Y_* \in \mathfrak{X}(N)$  (with respect to the submersion described in Section 4) we have

$$\begin{aligned} \nabla J(Y, X) &= \nabla_Y(JX) - J(\nabla_Y X) = \nabla_{Y_*}^*(J_*(X_*))^* - \frac{1}{2}d\theta(Y, JX)\xi \\ &\quad + T(Y, JX) - J\left(\nabla_{Y_*}^*(X_*)^* - \frac{1}{2}d\theta(Y, X)\xi + T(Y, X)\right) \\ &= -\frac{1}{2}sh(JY, JX)ff'H - gg'h(Y, JX)H - \frac{1}{2}sh(JY, X)ff'H + h(X, Y)gg'JH \\ &= h(X, Y)\left(\epsilon gg' - \frac{1}{2}sf\right)JH + h(JY, X)\left(gg' - \epsilon \frac{1}{2}sf\right)H. \end{aligned}$$

Hence

$$(5.4) \quad \nabla J(JX, JY) = \nabla J(X, Y).$$

Since the distribution  $\mathcal{D}$  is totally geodesic and two-dimensional it is clear that  $\nabla J(X, Y) = 0$  if  $X, Y \in \Gamma(\mathcal{D})$ . Now we shall show that

$$\nabla J(JX, JH) = \nabla J(X, H) \quad \text{for } X \in \mathcal{D}^\perp.$$

It is easy to show that  $\nabla_X H = \nabla_H X = \frac{g'}{g}X$  and

$$(5.5) \quad \nabla_X(JH) = \epsilon \nabla_X\left(\frac{1}{f}\xi\right) = \epsilon \frac{1}{f}T(X) = \epsilon \frac{sf}{2g^2}JX.$$

On the other hand,

$$(5.6) \quad \nabla_X(JH) = \nabla J(X, H) + J(\nabla_X H) = \nabla J(X, H) + \frac{g'}{g}J(X).$$

Thus  $\nabla J(X, H) = \left(\epsilon \frac{sf}{2g^2} - \frac{g'}{g}\right)JX$ . On the other hand,

$$\nabla_X(JJH) = -\frac{g'}{g}X = \nabla_X J(JH) + J(\nabla_X JH) = \nabla J(X, JH) - \epsilon \frac{sf}{2g^2}X.$$

Consequently,

$$\nabla J(X, JH) = \left(\epsilon \frac{sf}{2g^2} - \frac{g'}{g}\right)X.$$

It follows that

$$(5.7) \quad \nabla J(JX, JH) = \nabla J(X, H).$$

Similarly

$$(5.8) \quad \nabla_H(JX) = \nabla_{JX}H = \frac{g'}{g}JX = \nabla J(H, X) + J(\nabla_H X) = \nabla J(H, X) + \frac{g'}{g}J$$

and  $\nabla J(H, X) = 0$ . Let us recall that  $\nabla_X(\frac{1}{f}\xi) = \frac{sf}{2g^2}JX$ . Thus we have

$$(5.9) \quad \nabla J(JH, X) = \nabla_{JH}JX - J(\nabla_{JH}X) = \epsilon \left( \nabla_{JX} \left( \frac{1}{f}\xi \right) - J \left( \nabla_X \left( \frac{1}{f}\xi \right) \right) \right) = 0.$$

Thus

$$(5.10) \quad \nabla J(JH, JX) = \nabla J(H, X)$$

and consequently

$$\nabla J(Y, Z) = \nabla J(JY, JZ)$$

for all  $Y, Z \in \mathfrak{X}(M)$ , which means that  $(M, g, J_\epsilon)$  is a Hermitian manifold.

Let  $U, V, W \in \mathcal{V}$  and  $g(U, \xi) = g(V, \xi) = g(W, \xi) = 0$ . Then

$$R(U, V, \xi, W) = \hat{R}(U, V, \xi, W) - g(T(U, \xi), T(V, W)) + g(T(V, \xi), T(U, W)) = 0.$$

Analogously

$$(5.11) \quad R(U, V, \xi, H) = g(\nabla_V T(U, \xi), H) - g(\nabla_U T(V, \xi), H).$$

Note that  $T(U, \xi) = 0$ . Hence

$$0 = g(\nabla_V T(U, \xi), H) + g(T(\nabla_V U, \xi), H) + g(T(U, \xi), \nabla_V H).$$

Consequently,

$$(5.12) \quad g(\nabla_V T(U, \xi), H) - g(\nabla_U T(V, \xi), H) = g(T([U, V], \xi), H) = 0.$$

From the O'Neill formulae it also follows that

$$R(JH, U, V, JH) = 0$$

if  $g(U, V) = 0$ , and

$$(5.13) \quad R(JH, U, U, JH) = \frac{s^2 f^2}{4g^4} - \frac{f' g'}{f g}$$

for a unit vector field  $U$  as above. Note also that the distribution  $\mathcal{D}$  spanned by the vector fields  $\xi, H$  is totally geodesic. Consequently, if  $X, Y, Z \in \Gamma(\mathcal{D})$  and  $V$  is as above then

$$(5.14) \quad R(X, Y, Z, V) = 0.$$

Note also that

$$(5.15) \quad R(H, \frac{1}{f}\xi, \frac{1}{f}\xi, H) = -\frac{f''}{f}$$

and

$$(5.16) \quad R(H, X, X, H) = -\frac{g''}{g}$$

for  $X \in \Gamma(\mathcal{D}^\perp)$  and  $\|X\| = 1$ .

**6. Eigenvalues of the Ricci tensor.** Let us assume that  $(N, h)$  is a  $2(n-1)$ -dimensional Kähler-Einstein manifold of scalar curvature  $4(n-1)\epsilon$  where  $\epsilon \in \{-1, 0, 1\}$ . Using the results in Sections 3 and 4 we obtain the following formulae for the eigenvalues of the Ricci tensor  $\rho$  of  $(U_{s,f,g}, g_{f,g})$ :

$$(6.1) \quad \begin{aligned} \lambda_0 &= -2(n-1)\frac{g''}{g} - \frac{f''}{f}, \\ \lambda_1 &= -\frac{f''}{f} + 2(n-1)\left(\frac{s^2 f^2}{4g^4} - \frac{f'g'}{fg}\right), \\ \lambda_2 &= -\frac{g''}{g} + \left(\frac{s^2 f^2}{4g^4} - \frac{f'g'}{fg}\right) + \frac{2\epsilon}{g^2} - \frac{3s^2 f^2}{4g^4} - (2n-3)\frac{(g')^2}{g^2}. \end{aligned}$$

Since  $\lambda_0 = \lambda_1$  we obtain

$$(6.2) \quad \frac{g''}{g} + \frac{s^2 f^2}{4g^4} - \frac{f'g'}{fg} = 0.$$

Hence

$$(6.3) \quad f = \pm \frac{gg'}{\sqrt{\frac{s^2}{4} + Ag^2}},$$

where  $A \in \mathbb{R}$ . Using a homothety we can assume that  $A \in \{-1, 0, 1\}$ . Under this condition the Ricci tensor  $\rho$  has two eigenvalues  $\lambda = -2(n-1)\frac{g''}{g} - \frac{f''}{f}$  and  $\mu = -\frac{g''}{g} - \frac{f'g'}{fg} + \frac{2\epsilon}{g^2} - \frac{s^2 f^2}{2g^4} - (2n-3)\frac{(g')^2}{g^2}$ . We shall discuss separately the conditions  $A = 0$  and  $|A| = 1$ . Let us write  $h^2 = \frac{s^2}{4} + Ag^2$ . Then  $g = \sqrt{[(\frac{s}{2})^2 - h^2]}$  and  $\text{im } h \subset (-\frac{s}{2}, \frac{s}{2})$  if  $A = -1$  and  $\text{im } h \subset (\frac{s}{2}, \infty)$  if  $A = 1$ . Note that  $h' \neq 0$ . Let us assume that  $h$  is an increasing function. Then

$$f = \pm \frac{gg'}{|h|} = \pm \frac{hh'A}{|h|} = h'.$$

We also have

$$g'' = \frac{A((h')^2 + hh'')}{g} - \frac{hh'Ag'}{g^2} = \frac{A((h')^2 + hh'')}{g} - \frac{h^2(h')^2}{g^3}.$$

Thus

$$\frac{g''}{g} = \frac{A((h')^2 + hh'')}{g^2} - \frac{h^2(h')^2}{g^4} = -\frac{(h')^2 + hh''}{r^2 - h^2} - \frac{h^2(h')^2}{(r^2 - h^2)^2}.$$

Since  $h' > 0$  it follows that  $t$  is a smooth function of  $h$ , i.e.  $t = t(h)$  and  $\frac{dt}{dh} = \frac{1}{h'}$ . Let  $z$  be a function such that  $h' = \sqrt{z(h)}$ , i.e.  $z(h) = h'(t(h))^2$ . Then

$$f' = h'' = \frac{1}{2}z'(h).$$

Consequently,

$$\frac{g''}{g} = -\frac{z(h) + \frac{1}{2}hz'(h)}{r^2 - h^2} - \frac{h^2z(h)}{(r^2 - h^2)^2}.$$

Analogously

$$\left(\frac{g'}{g}\right)^2 = \left(\frac{hh'A}{g^2}\right)^2 = \frac{h^2z(h)}{(r^2 - h^2)^2},$$

and

$$\frac{f'g'}{fg} = \frac{\frac{1}{2}z'(h)hh'A}{h'g^2} = \frac{\frac{1}{2}z'(h)hA}{-A(r^2 - h^2)} = -\frac{\frac{1}{2}z'(h)h}{r^2 - h^2}.$$

We also have

$$\frac{s^2f^2}{2g^4} = 2r^2 \frac{z(h)}{(r^2 - h^2)^2}.$$

Since

$$\mu = -\frac{g''}{g} - \frac{f'g'}{fg} + \frac{2\epsilon}{g^2} - \frac{s^2f^2}{2g^4} - (2n-3)\frac{(g')^2}{g^2},$$

we get

$$\mu = \frac{hz'(h)}{r^2 - h^2} - \frac{z(h)}{(r^2 - h^2)^2}(r^2 + (2n-3)h^2) - \frac{2\epsilon A}{r^2 - h^2}.$$

We shall use the following:

**Proposition 2.** *Let  $\mathcal{D}$  be a distribution on  $U_{s,f,g}$  spanned by the fields  $\{\theta^\sharp, H\}$ . Then  $\mathcal{D}$  is a totally geodesic foliation with respect to the metric  $g_{f,g}$ . The distribution  $\mathcal{D}^\perp$  is umbilical with the mean curvature normal  $\xi = -\nabla \ln g$ . Let  $\lambda, \mu$  be the eigenvalues of the Ricci tensor  $S$  of  $g_{f,g}$  corresponding to the eigendistributions  $\mathcal{D}, \mathcal{D}^\perp$  respectively. Then the following conditions are equivalent:*

- (a) *There exists  $E \in \mathbb{R}$  such that  $\lambda - \mu = Eg^2$ ,*
- (b) *There exist  $C, D \in \mathbb{R}$  such that  $\mu = Cg^2 + D$ ,*
- (c)  *$\lambda - 2\mu$  is constant,*
- (d)  *$(U_{s,f,g}, g_{f,g})$  is a Hermitian Gray manifold.*

*Proof.* The first part of the assertion is a consequence of [J-2]. Note that  $\nabla \lambda = H\lambda H, \nabla \mu = H\mu H$ . Consequently,  $\text{tr}_g \nabla S = \frac{1}{2}\nabla \tau = (H\lambda + (n-1)H\mu)H$ . On the other hand, one can easily check that  $\text{tr}_g \nabla S = 2(n-1)(\mu - \lambda)\xi + H\lambda H$ . Thus

$$\frac{\nabla \mu}{2(\lambda - \mu)} = \nabla \ln g.$$

Now we prove that (a)  $\Rightarrow$  (b). If (a) holds then  $\nabla\mu = 2Eg^2 \frac{\nabla g}{g} = E\nabla g^2$ . Thus  $\nabla(\mu + Eg^2) = 0$ , which implies (b).

(b)  $\Rightarrow$  (a). We have

$$\frac{\nabla g}{g} = \frac{\nabla\mu}{2(\lambda - \mu)} = \frac{Cg\nabla g}{\lambda - \mu},$$

and consequently  $\nabla g(\frac{Cg^2 - (\lambda - \mu)}{g(\lambda - \mu)}) = 0$ , which is equivalent to (a).

(a)  $\Rightarrow$  (c). We have  $\lambda - \mu = Eg^2$  and consequently  $\nabla\mu = 2Eg\nabla g = E\nabla g^2$ . Thus  $\nabla\lambda = \nabla(\mu + Eg^2) = 2E\nabla g^2$  and  $\nabla\lambda - 2\nabla\mu = 0$ , which gives (c).

(c)  $\Rightarrow$  (a). If  $\nabla\lambda = 2\nabla\mu$  then  $\nabla\lambda = 4(\lambda - \mu)\frac{\nabla g}{g}$ . Consequently,  $\nabla\lambda - \nabla\mu = 2(\lambda - \mu)\frac{\nabla g}{g}$  and  $\nabla \ln|\lambda - \mu| = 2\frac{\nabla g}{g} = 2\nabla \ln g$ , which means that  $\nabla \ln|\lambda - \mu|g^{-2} = 0$ . It follows that  $\ln \frac{|\lambda - \mu|}{g^2} = C$  for some  $C \in \mathbb{R}$ , which is equivalent to (a).

(d)  $\Leftrightarrow$  (c). This equivalence follows from Proposition 1.  $\diamond$

*Remark.* From the condition (c) in Proposition 2 it is clear that the eigenvalues  $\lambda, \mu$  of the Ricci tensor are everywhere different in the case of a  $\mathbb{CP}^1$ -bundle and coincide at an exactly one point if  $M = \mathbb{CP}^n$ .

**7. Solutions of the linear ODE related to Gray manifolds.** Using Prop. 2 we see that the equation characterizing our Gray manifold is

$$\frac{hz'(h)}{r^2 - h^2} - \frac{z(h)}{(r^2 - h^2)^2}(r^2 + (2n - 3)h^2) = \frac{2\epsilon A}{r^2 - h^2} + C(r^2 - h^2) + D,$$

or

$$(7.1) \quad z'(h) - \frac{z(h)}{h(r^2 - h^2)}(r^2 + (2n - 3)h^2) = \frac{2\epsilon A}{h} + C\frac{(r^2 - h^2)^2}{h} + D\frac{r^2 - h^2}{h},$$

where  $C, D$  are any real numbers. We are looking for the solutions of (7.1) satisfying the following boundary conditions: if  $A = -1$  then there exist real numbers  $x, y$  such that  $-r < x < y < r$  and if  $A = 1$  then  $r < x < y$  and in both cases  $z(x) = 0, z'(x) = 2, z(y) = 0, z'(y) = -2$  and  $z(t) > 0$  if  $t \in (x, y)$ .

**The case  $s \neq 0, A \in \{-1, 1\}$ .** Let us write  $z(t) = z_0(\frac{t}{r})$ . Then it is easy to check that the function  $z_0$  satisfies the equation

$$(7.2) \quad z'_0(h) - \frac{z_0(h)}{h(1 - h^2)}(1 + (2n - 3)h^2) = \frac{2\eta}{h} + r^4 C \frac{(1 - h^2)^2}{h} + r^2 D \frac{1 - h^2}{h},$$

where  $\eta = \epsilon A$ , and the boundary conditions, if  $A = -1$  then there exist real numbers  $x, y$  such that  $-1 < x < y < 1$  and if  $A = 1$  then  $1 < x < y$  and in both cases  $z_0(x) = 0, z'_0(x) = 2r, z_0(y) = 0, z'_0(y) = -2r$  and  $z(t) > 0$  if  $t \in (x, y)$ .

We have  $z_0(t) = F(t)\frac{t}{(1 - t^2)^{n-1}}$  and the function  $F$  satisfies the equation:

$$(7.3) \quad F'(t) = 2\eta \frac{(1 - t^2)^{n-1}}{t^2} + C \frac{(1 - t^2)^{n+1}}{t^2} + D \frac{(1 - t^2)^n}{t^2},$$

where we denote  $r^4 C, r^2 D$  again by  $C, D$ . Hence

$$(7.4) \quad F(t) = -\frac{2\eta}{t} + 2\eta \sum_{k=1}^{n-1} (-1)^k C_{n-1}^k \frac{t^{2k-1}}{2k-1} \\ - \frac{C}{t} + C \sum_{k=1}^{n+1} (-1)^k C_{n+1}^k \frac{t^{2k-1}}{2k-1} - \frac{D}{t} + D \sum_{k=1}^n (-1)^k C_n^k \frac{t^{2k-1}}{2k-1} + E,$$

where  $E \in \mathbb{R}$  and  $C_n^k = \frac{n!}{k!(n-k)!}$ . From the boundary conditions we obtain

$$(7.5) \quad C = \frac{2(-r + \eta x - \eta y + rxy)}{(x^2 - 1)(-x + y + xy^2 - y^3)},$$

$$D = \frac{2(\eta(x^3 - x^2y + x(y^2 - 2) - y(y^2 - 2) + r(1 + x^3y - x^2y^2 + xy(y^2 - 2)))}{(x^2 - 1)(y^2 - 1)(x - y)},$$

if  $x \neq -y$ . Consequently, the condition on  $x, y$  such that  $x \neq -y$  is as follows:

$$(7.6) \quad -\frac{2\eta}{x} + 2\eta \sum_{k=1}^{n-1} (-1)^k C_{n-1}^k \frac{x^{2k-1}}{2k-1} - \frac{C}{x} + C \sum_{k=1}^{n+1} (-1)^k C_{n+1}^k \frac{x^{2k-1}}{2k-1}$$

$$- \frac{D}{x} + D \sum_{k=1}^n (-1)^k C_n^k \frac{x^{2k-1}}{2k-1} + \frac{2\eta}{y} - 2\eta \sum_{k=1}^{n-1} (-1)^k C_{n-1}^k \frac{y^{2k-1}}{2k-1}$$

$$+ \frac{C}{t} - C \sum_{k=1}^{n+1} (-1)^k C_{n+1}^k \frac{y^{2k-1}}{2k-1} + \frac{D}{y} - D \sum_{k=1}^n (-1)^k C_n^k \frac{y^{2k-1}}{2k-1} = 0,$$

where  $C, D$  are given by formulae (7.5) and

$$-E = -\frac{2\eta}{x} + 2\eta \sum_{k=1}^{n-1} (-1)^k C_{n-1}^k \frac{x^{2k-1}}{2k-1} - \frac{C}{x} + C \sum_{k=1}^{n+1} (-1)^k C_{n+1}^k \frac{x^{2k-1}}{2k-1}$$

$$- \frac{D}{x} + D \sum_{k=1}^n (-1)^k C_n^k \frac{x^{2k-1}}{2k-1}.$$

We are looking for  $(x, y)$  lying on the algebraic curve given by (7.6). We show that the function given by a solution  $(x, y)$  of these equations really gives the  $\mathcal{AC}^\perp$  Hermitian manifold at least if  $x > 0$  or  $y < 0$ . Let us assume that either  $0 < x < y < 1$ ,  $1 < x < y$  or  $-1 < x < y < 0$  and there exist two points  $p, q \in (x, y)$  such that  $F'(p) = F'(q) = 0$ . Then

$$0 = 2\eta + C(1 - p^2)^2 + D(1 - p^2),$$

$$0 = 2\eta + C(1 - q^2)^2 + D(1 - q^2).$$

Thus

$$C = C(p, q) = \frac{2\eta}{(p^2 - 1)(q^2 - 1)}, \quad D = D(p, q) = \frac{2\eta(-2 + p^2 + q^2)}{(p^2 - 1)(q^2 - 1)}.$$

It follows that for a given  $p$  such that  $F'(p) = 0$  there exists at least one  $q \in (x, y)$ ,  $q \neq p$  such that  $F'(q) = 0$  since the function  $I \ni q \mapsto C(p, q) \in \mathbb{R}$  where  $I = (-1, 0)$ ,  $I = (0, 1)$  or  $I = (1, \infty)$  is injective.

If  $x = -y$  then  $E = 0$ . Note that if  $E = 0$  then  $z_0$  is an even function. Now we consider the case  $A = -1$ ,  $-1 < x < 0$  and  $y = -x$ . Then

$$C = \frac{2rx - 2\eta - D(1 - x^2)}{(1 - x^2)^2}.$$



We shall find  $D$  such that  $z_0(x) = 0$ , which means  $F(x) = 0$ . We have

$$\begin{aligned} & -\frac{2\eta}{x} + 2\eta \sum_{k=1}^{n-1} (-1)^k C_{n-1}^k \frac{x^{2k-1}}{2k-1} - \frac{2rx - 2\eta - D(1-x^2)}{(1-x^2)^2 x} \\ & + \frac{2rx - 2\eta - D(1-x^2)}{(1-x^2)^2} \sum_{k=1}^{n+1} (-1)^k C_{n+1}^k \frac{x^{2k-1}}{2k-1} \\ & - \frac{D}{x} + D \sum_{k=1}^n (-1)^k C_n^k \frac{x^{2k-1}}{2k-1} = 0. \end{aligned}$$

Hence

$$\begin{aligned} & D \left( \frac{1}{1-x^2} - \frac{1}{1-x^2} \sum_{k=1}^{n+1} (-1)^k C_{n+1}^k \frac{x^{2k-1}}{2k-1} - 1 + \sum_{k=1}^n (-1)^k C_n^k \frac{x^{2k-1}}{2k-1} \right) \\ & = 2\eta - 2\eta \sum_{k=1}^{n-1} (-1)^k C_{n-1}^k \frac{x^{2k-1}}{2k-1} + \frac{2rx - 2\eta}{(1-x^2)^2} - \frac{2rx - 2\eta}{(1-x^2)^2} \sum_{k=1}^{n+1} (-1)^k C_{n+1}^k \frac{x^{2k-1}}{2k-1}. \end{aligned}$$

Thus

$$\begin{aligned} (7.7) \quad & D \left( 2x^2 - C_{n+1}^2 \frac{x^4}{3} + C_n^2 \frac{x^4}{3} + nx^4 + \phi_5(x) \right) \\ & = \frac{1}{1-x^2} (2rx + 2\eta(n-3)x^2 + 2r(n+1)x^3 + \psi_4(x)) \end{aligned}$$

and

$$(7.8) \quad D \left( 1 + \frac{n}{3}x^2 + \phi_3(x) \right) = \frac{1}{1-x^2} \left( \frac{r}{x} + \eta(n-3) + r(n+1)x + \psi_2(x) \right)$$

and

$$(7.9) \quad D = \frac{1}{1-x^2} \left( \frac{r}{x} + \eta(n-3) + 2r \left( \frac{5}{6}n + 1 \right) x + \psi_2(x) \right).$$

Hence

$$(7.10) \quad C = -\frac{r}{x} - 2\eta - \eta(n-3) - 2r \left( \frac{5}{6}n + 1 \right) x + \phi_2(x),$$

where  $f_k(x)$  denotes an analytic function of the form  $f_k(x) = x^k(a_0 + a_1x + a_2x^2 + \dots)$  convergent in a neighborhood of  $x = 0$ . It follows that

$$\begin{aligned} (7.11) \quad & z_x(0) = -2\eta - C - D = -2\eta + \frac{r}{x} + 2\eta + \eta(n-3) + 2r \left( \frac{5}{6}n + 1 \right) x \\ & - \phi_2(x) - \frac{r}{x} - \eta(n-3) - 2r \left( \frac{5}{6}n + 1 \right) x - rx + \gamma_2(x) = -rx + \alpha_2(x). \end{aligned}$$

Consequently,  $z_x(0) > 0$  for sufficiently small  $|x|$  where  $x \in (-1, 0)$ . Now we shall show that the function  $z_x$  satisfies the condition  $z_x(t) > 0$  for  $t \in (x, -x)$ . To this end we shall look at  $F'_x(t) = \frac{(1-t^2)^{n-1}}{t^2}(2\eta + C(1-t^2) + D(1-t^2))$  where  $C, D$  are given by (7.9), (7.10). We shall prove that for small  $x$  the function  $F'_x(t)$  has no zeros in the interval  $(x, 0)$ . Since  $F(x) = 0$ ,  $F'(x)\frac{x}{(1-x^2)^{n-1}} = 2r > 0$  and  $\lim_{t \rightarrow 0^-} F(t) = -\infty$  it follows that if  $F(t)$  had a zero in  $(x, 0)$  then  $F(t)$  would have at least two zeros in  $(x, 0)$ .

Note that  $F'(t) = 0$  if and only if (we shall assume that  $C \neq 0$ )

$$(7.12) \quad C\left(\alpha^2 + \frac{D}{C}\alpha + \frac{2\eta}{C}\right) = 0,$$

where  $\alpha = 1 - t^2$ . Note that

$$\frac{D}{C} = -\frac{1 + \eta(n-3)\frac{x}{r} + \psi_2(x)}{1 + \eta(n-1)\frac{x}{r} + \phi_2(x)}.$$

Consequently,

$$\frac{D}{C} = -1 + \frac{2\eta}{r}x + \gamma_2(x).$$

Analogously  $8\frac{\eta}{C} = 8\eta\frac{x}{r} + \beta_2(x)$ . It follows that

$$\alpha = \frac{1}{2}\left(1 - \frac{2\eta}{r}x - \gamma_2(x) \pm \sqrt{1 + \frac{4\eta}{r}x + o_2(x)}\right)$$

are the solutions of (7.12). Hence one of the roots is  $\alpha = 1 - t^2 = -4\frac{\eta}{r}x + k_2(x)$  and then  $t^2 = 1 + 4\frac{\eta}{r}x - k_2(x)$ . For a small  $x$  this  $|t|$  is close to 1. Thus  $F'(t)$  can have at most one zero in  $(-x, 0)$ . It follows that  $F(t)$  is different from 0 in  $(x, 0)$  and consequently  $z_x(t)$  is positive on the whole of  $(x, -x)$  for small  $x \in (-1, 0)$ . This means that in the case of  $|A| = 1$  we have infinitely many examples of compact bi-Hermitian Gray manifolds of any dimension  $2n$  corresponding to a given compact Kähler Einstein manifold  $(N, h)$  and a rational number  $s$  described in Section 3. They are holomorphic  $\mathbb{CP}^1$ -bundles over  $N$ .

Now we consider the case where  $M = \mathbb{CP}^n$ . Then  $N = \mathbb{CP}^{n-1}$ ,  $s = \frac{2}{n}$ ,  $k = 1$ . One of the zeros of  $z_0$  is 1 and let  $y$  be the other zero such that  $z_0 > 0$  on the interval  $(y, 1)$  or  $(1, y)$ . Note that  $y < 1$  if  $\eta = A = -1$ , and  $y > 1$  if  $A = 1$ . It is easily seen, that  $z'_0(1) = \frac{2A}{n} = As$ . Since  $z_0(y) = 0$  and  $z'_0(y) = -sA = -\frac{2A}{n}$  we get from (7.2):

$$(7.13) \quad 2A + C(1 - y^2)^2 + D(1 - y^2) = -\frac{2Ay}{n}.$$

Let us assume that  $z'_0(x) = 0$  for  $x \in (y, 1)$  (or  $x \in (1, y)$  respectively). The functions  $z_0(t), F(t)$  have the same set of zeros in the interval  $(0, +\infty)$ . We show that the only zeros of  $F(t)$  in  $[y, 1]$  (or  $[1, y]$ ) are 1,  $y$ . Since  $F(y) = F(1) = 0$  it is enough to show that  $F'(t)$  has at most one zero in  $(y, 1)$  ( $(1, y)$  respectively). Let  $F'(x) = 0$ . Then

$$(7.14) \quad -2A = C(1 - x^2)^2 + D(1 - x^2),$$

and

$$(7.15) \quad \begin{aligned} C &= C(x, y) = \frac{2A(nx^2 - y + x^2y - ny^2)}{n(1 - x^2)(x^2 - y^2)(1 - y^2)}, \\ D &= D(x, y) = \frac{2A(-2nx^2 + nx^4 + y - 2x^2y + 2ny^2 - ny^4)}{n(y^2 - 1)(x^2 - y^2)(x^2 - 1)}. \end{aligned}$$

The derivative of  $C = C(x, y)$  as a function of  $x$  is

$$\frac{\partial C}{\partial x} = \frac{4Ax((x^2 - 1)^2y + n(x^2 - y^2)^2)}{n(x^2 - 1)^2(x^2 - y^2)^2(1 - y^2)},$$

and is always less than 0 if  $A = 1$  and greater than 0 if  $A = -1$  for  $y > 0$ . It follows that the function  $(y, 1) \ni x \mapsto C(x, y) \in \mathbb{R}$  is injective. It follows that  $F'(t)$  can have at most one zero in  $(y, 1)$ . Consequently, the function  $z_0$  is positive on  $(y, 1)$  (or  $(1, y)$ ) if  $z_0$  is a solution of (7.2) satisfying the boundary conditions  $z_0(y) = 0, z_0'(y) = -\frac{2A}{n}, z_0(1) = 0$ . Consequently, we obtain two families,  $(\mathbb{CP}^n, g_x), x \in (0, 1) \cup (1, \infty)$ , of Hermitian  $\mathcal{AC}^\perp$  metrics on  $\mathbb{CP}^n$ . Note that the solutions exist also for  $x < 0$  (compare for example the case  $n = 2$  [J-3]).

**The case  $A = 0$ .** This case can also be deduced from [A-C-G-T]. For a convenience of the reader we give a shorter proof here. Now we shall consider the case  $A = 0$ , which corresponds to the Kähler manifold  $(M, g_{f,g})$ . In this case we have  $f = \frac{2gg'}{s}$  and

$$\mu = -\frac{g''}{g} + \frac{2\epsilon}{g^2} - 2n\frac{(g')^2}{g^2},$$

and we obtain the equation

$$(7.16) \quad z'(g) - 2n\frac{z(g)}{g} = \frac{2\epsilon}{g} - Cg^3 - Dg,$$

It follows that

$$z(g) = \frac{\epsilon}{n} - C\frac{g^4}{2n+4} - D\frac{g^2}{2n+2} + \frac{E}{g^{2n}}.$$

We are looking for the solution  $z(t)$  satisfying for certain points  $0 < x < y$  the initial conditions:

$$(7.17) \quad z(x) = z(y) = 0, \quad xz'(x) = s, \quad yz'(y) = -s,$$

and such that  $z(t) > 0$  for  $t \in (x, y)$ . Hence

$$\begin{aligned} C &= \frac{-2\epsilon + s - Dx^2}{x^4}, E = -\frac{x^{2n}(4\epsilon + 4\epsilon n + ns + n^2s - Dnx^2)}{2n(1+n)(2+n)}, \\ C &= \frac{-2\epsilon - s + Dy^2}{y^4}, E = \frac{y^{2n}(-4\epsilon - 4\epsilon n + ns + n^2s + Dny^2)}{2n(1+n)(2+n)}. \end{aligned}$$

It follows that

$$D = \frac{2\epsilon x^4 + sx^4 - 2\epsilon y^4 + sy^4}{(x^2 - y^2)x^2y^2}$$

$$D = \frac{(1+n)(4\epsilon x^{2n} + nsx^{2n} - 4\epsilon y^{2n} + nsy^{2n})}{n(x^{2+2n} - y^{2+2n})}.$$

Hence there exists a function  $z$  satisfying (7.17) if and only if

$$(7.18) \quad n(s+2\epsilon)x^{2n+6} - n(s-2\epsilon)y^{2n+6} - [(6n+4)\epsilon + n^2s]x^{2n+2}y^4 \\ - [(6n+4)\epsilon - n^2s]y^{2n+2}x^4 + [(4\epsilon(n+1) + n(n+1)s)]x^{2n+4}y^2 \\ + [(4\epsilon(n+1) - n(n+1)s)]y^{2n+4}x^2 = 0$$

for certain  $0 < x < y$ . This equation is equivalent to

$$(7.19) \quad -n(s-2\epsilon)c^{2n+6} + [(4\epsilon(n+1) - n(n+1)s)]c^{2n+4} - [(6n+4)\epsilon - n^2s]c^{2n+2} \\ - [(6n+4)\epsilon + n^2s]c^4 + [4\epsilon(n+1) + n(n+1)s]c^2 + n(s+2\epsilon) = 0,$$

where  $c = \frac{y}{x} > 1$ . Let

$$\phi(c) = -n(s-2\epsilon)c^{2n+6} + [(4\epsilon(n+1) - n(n+1)s)]c^{2n+4} - [(6n+4)\epsilon - n^2s]c^{2n+2} \\ - [(6n+4)\epsilon + n^2s]c^4 + [4\epsilon(n+1) + n(n+1)s]c^2 + n(s+2\epsilon).$$

Then  $\phi(1) = 0$  and  $\phi'(1) = -8ns(n+1) < 0$ . It follows that if  $s - 2\epsilon < 0$  then there exists a root  $c_0 \in (1, \infty)$  of equation (7.19). Note that  $\phi(c) = \epsilon\Phi(c) + s\Psi(c)$ , where  $\Phi(c) = 2nc^{2n+6} + 4(n+1)(c^{2n+4} + c^2) - (6n+4)(c^4 + c^2) + 2n$  and  $\Psi(c) = -n(c^{2n+6} + (n+1)(c^{2n+4} - c^2) - n(c^{2n+2} - c^4) - n)$ . Note that for  $c > 1$ ,  $\Phi(c) > 0$  and  $\Psi(c) < 0$ . This follows from the inequalities  $(n+1)(c^{2n+4} - c^2) > n(c^{2n+2} - c^4)$  and  $2n(c^{2n+6} + 1) + 4(n+1)(c^{2n+4} + c^2) > (6n+4)(c^4 + c^2)$  valid for  $c > 1$ . Hence for  $\epsilon \in \{-1, 0\}$  we get  $\phi(c) < 0$  for  $c > 1$ . On the other hand, it is easy to see that  $\Phi(c) + 2\Psi(c) < 0$  if  $c > 1$ . It follows that the polynomial  $\phi$  has a real root  $c_0 > 1$  if and only if  $\epsilon = 1$  and  $s < 2$ . We shall show that the function

$$z(t) = \frac{\epsilon}{n} - C \frac{t^4}{2n+4} - D \frac{t^2}{2n+2} + \frac{E}{t^{2n}},$$

is positive on  $(x, c_0x)$  for any  $x > 0$ . Note that the derivative of  $z(t)$  can have at most two positive zeros and consequently if  $z(t)$  satisfies the conditions (7.17) then it is positive for  $t \in (x, y)$  where  $y = c_0x$ . Note also that for a given  $c_0$  the solutions given by  $(x, c_0x)$  give homothetic metrics for all  $x > 0$ . Since  $s = \frac{2k}{q} < 2$  it follows that  $k = 1, 2, \dots, q-1$  and for these values of  $k$  there exists a Kähler Gray bi-Hermitian metric on an appropriate  $\mathbb{CP}^1$ -bundle over a compact Kähler Einstein manifold of positive scalar curvature. Note that every such Kähler metric satisfies the equation (see [J-1])

$$\nabla_X \rho(Y, Z) = \frac{1}{2\dim M + 4} (g(X, Y)Z\tau + g(X, Z)Y\tau + 2g(Y, Z)X\tau \\ - g(JX, Y)(JZ)\tau - g(JX, Z)(JY)\tau).$$

**The case  $s = 0$ .** We shall assume that  $g$  is not constant. Then  $f = g'$  and

$$\mu = -2\frac{g''}{g} - (2n-3)\frac{(g')^2}{g^2} + \frac{2\epsilon}{g^2}.$$

Thus we obtain the equation

$$z'(g) + (2n-3)\frac{z(g)}{g} = \frac{2\epsilon}{g} - Cg^3 - Dg.$$

It follows that

$$z(t) = \frac{2\epsilon}{2n-3} - C\frac{t^4}{2n+1} - D\frac{t^2}{2n-1} + E\frac{1}{t^{2n-3}}.$$

Again the derivative of  $z$  can have at most two positive roots. Consequently, in the case  $\epsilon \neq 0$  there exists a nontrivial solution satisfying the initial conditions

$$(7.20) \quad z(x) = z(y) = 0, \quad z'(x) = 2, \quad z'(y) = -2,$$

for  $0 < x < y$  if

$$x = \epsilon \frac{4(1-c)(c^{2n+1}-1)(2n-1) + 2(-1+c-c^2+c^3)(c^{2n-1}-1)(2n+1)}{(2n-3)c((c^{2n+1}-1)(2n-1) + (1-c+c^2)(1-c^{2n-1})(2n+1))}$$

is positive for some  $c \in (1, +\infty)$ . For  $n = 2, 3$  it can happen only if  $\epsilon = 1$ , and for  $n \in \{4, 5, \dots\}$ ,  $x$  is positive for some  $c > 1$  for both values of  $\epsilon \in \{-1, 1\}$ . Note that the polynomial  $4(1-c)(c^{2n+1}-1)(2n-1) + 2(-1+c-c^2+c^3)(c^{2n-1}-1)(2n+1)$  is negative for  $c > 1$ . So the sign of  $\epsilon x$  is opposite to the sign of  $Q(c) = c((c^{2n+1}-1)(2n-1) + (1-c+c^2)(1-c^{2n-1})(2n+1))$ . If  $n \in \{2, 3\}$  then  $Q$  is strictly negative for  $c > 1$ . For  $n = 2$  this is proved in [J-2],[J-3] and for  $n = 3$  we have  $Q(c) = -(c-1)^5 c(2c^2 + 3c + 2)$ . Note also that  $Q(1) = Q'(1) = Q''(1) = 0$  and  $Q^{(3)}(1) = 2(1+2n)(2n^2 - 7n + 3)$ . Hence  $Q(c) > 0$  for  $c > 1$  close to 1 if  $n > 3$ . Since  $\lim_{c \rightarrow +\infty} Q(c) = -\infty$  it follows that  $Q$  changes sign and consequently there exist solutions for both  $\epsilon = 1$  and  $\epsilon = -1$ . This means that if  $(M, g, J)$  is a compact Kähler Einstein manifold of dimension  $\geq 6$  and of non-zero scalar curvature then there exists a family of compact bi-Hermitian Gray metrics on the manifold  $\mathbb{CP}^1 \times M$  with a warped product metric. If  $\dim M \leq 4$  then the appropriate metric exists only if the scalar curvature of the Kähler Einstein manifold  $M$  is positive.

In the case  $\epsilon = 0$  there exists a solution satisfying the boundary conditions if the polynomial  $F(c) = (1+c)(1-c^{2n+1})(2n-1) + (1+c^3)(c^{2n-1}-1)(2n+1)$  has a real root  $c > 1$ . Then  $n > 3$  again. In fact,  $F(1) = 0$ ,  $F'(1) = F''(1) = 0$  and  $F^{(3)}(1) = (n-3)(1-4n^2)$ . It follows that for  $n > 3$  the polynomial  $F$  takes negative values for  $c > 1$  close to 1. Since  $\lim_{c \rightarrow +\infty} F(c) = +\infty$  it follows that  $F$  has a real root  $c_0 > 1$ . If  $n = 3$  then  $F(c) = (c-1)^5(2c^3 + 5c^2 + 5c + 2)$  is obviously positive for  $c > 1$ . The case  $n = 2$  was considered in [J-3]. In that way we obtain Gray bi-Hermitian metrics on the products  $\mathbb{CP}^1 \times M$  where  $M$  is a Kähler Ricci flat manifold of dimension  $\geq 6$ .

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Institute of Mathematics  
Cracow University of Technology  
Warszawska 24  
31-155 Kraków, POLAND.  
E-mail address: [wjelon@pk.edu.pl](mailto:wjelon@pk.edu.pl)